Level sets and Composition operators on the Dirichlet space

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LEVEL SETS AND COMPOSITION OPERATORS ON THE DIRICHLET SPACE

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Abstract. We consider composition operators in the Dirichlet space of the unit disc in the plane. Various criteria on boundedness, compactness and Hilbert-Schmidt class membership are established. Some of these criteria are shown to be optimal.

1. Introduction

In this note we consider composition operators in the Dirichlet space of the unit disc. A comprehensive study of composition operators in function spaces and their spectral behavior could be found in [3, 10, 16]. See also [6, 7, 8, 12, 13, 17] for a treatment of some of the questions addressed in this paper.

Let $\mathbb{D}$ be the unit disc in the complex plane and let $\mathbb{T} = \partial\mathbb{D}$ be its boundary. We denote by $\mathcal{D}$ the classical Dirichlet space. This is the space of all analytic functions f on $\mathbb{D}$ such that

$$\mathcal{D}(f) := \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty,$$

where $dA(z) = dxdy/\pi$ stands for the normalized area measure in $\mathbb{D}$. We call $\mathcal{D}(f)$ the Dirichlet integral of f. The space $\mathcal{D}$ is endowed with the norm

$$\|f\|^2_{\mathcal{D}} := |f(0)|^2 + \mathcal{D}(f).$$

It is standard that a function $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n)\, z^n$, holomorphic on $\mathbb{D}$, belongs to $\mathcal{D}$ if and only if

$$\sum_{n\geq 0} |\widehat{f}(n)|^2 \,(1+n) < \infty,$$

and that this series defines an equivalent norm on $\mathcal{D}$.

Since the Dirichlet space is contained in the Hardy space $\mathrm{H}^2(\mathbb{D})$, every function $f \in \mathcal{D}$ has non-tangential limits f^* almost everywhere

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on $\mathbb{T}$. In this case, however, more can be said. Indeed, Beurling [2] showed that if $f \in \mathcal{D}$ then $f^*(\zeta) = \lim_{r\to 1} f(r\zeta)$ exists for $\zeta \in \mathbb{T}$ outside of a set of logarithmic capacity zero.

Let φ be a holomorphic self-map of $\mathbb{D}$. The composition operator C_φ on $\mathcal{D}$ is defined by

$$C_\varphi(f) = f \circ \varphi, \qquad f \in \mathcal{D}.$$

We are interested herein in describing the spectral properties of the composition operator C_φ, such as compactness and Hilbert-Schmidt class membership, in terms of the size of the level set of φ. For $s \in (0,1)$, the level set $E_\varphi(s)$ of φ is given by

$$E_\varphi(s) = \{\zeta \in \mathbb{T}\colon\ |\varphi(\zeta)| \geq s\}.$$

We give new characterizations of Hilbert-Schmidt class membership in the case of the Dirichlet space. We also establish the sharpness of these results.

2. A general criterion

For $\alpha > -1$, dA_α will denote the finite measure on $\mathbb{D}$ given by

$$dA_\alpha(z) := (1+\alpha)(1-|z|^2)^\alpha dA(z).$$

For $p \geq 1$ and $\alpha > -1$, the weighted Bergman space $\mathcal{A}^p_\alpha$ consists of the holomorphic functions f on $\mathbb{D}$ for which

$$\|f\|_{p,\alpha} := \Big[\int_{\mathbb{D}} |f(z)|^p \, dA_\alpha(z)\Big]^{1/p} < \infty.$$

We denote by $\mathcal{D}^p_\alpha$ the space consisting of analytic functions f on $\mathbb{D}$ such that

$$\|f\|^p_{\mathcal{D}^p_\alpha} := |f(0)|^p + \|f'\|^p_{p,\alpha} < \infty.$$

Appropriate choices of the parameter α give, with equivalent norm, all the standard holomorphic function spaces. Indeed, The Hardy space H^2 can be identified with $\mathcal{D}^2_1$. The classical Besov space is precisely $\mathcal{D}^p_{p-2}$, and if $p < \alpha + 1$, $\mathcal{D}^p_\alpha = \mathcal{A}^p_{\alpha-2}$. Finally, the classical Dirichlet space $\mathcal{D}$ is identical to $\mathcal{D}^2_0$.

We recall that, by the reproducing formula, one has

$$f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^{2+\alpha}} \, dA_\alpha(w), \qquad z \in \mathbb{D}, \tag{1}$$

for every $f \in \mathcal{A}^p_\alpha$ (see [16]).

Lemma 2.1. *Let $p \geq 1$ and let $\sigma > -1$. Then, there exists a constant C depending only on p and σ such that*

$$|f(z)|^p \;\; \leq \;\; C \int_{\mathbb{D}} \frac{|f(\lambda)|^p}{|1-\overline{\lambda}z|^{2+\sigma}}\, dA_\sigma(\lambda),$$

for every $f \in \mathcal{A}^p_\sigma$ and $z \in \mathbb{D}$.

Proof. By the above reproducing formula,

$$\frac{f(z)}{1-z\overline{w}} \;\; = \;\; \int_{\mathbb{D}} \frac{f(\lambda)}{1-\lambda\overline{w}}\, \frac{dA_\sigma(\lambda)}{(1-\overline{\lambda}z)^{2+\sigma}}, \qquad z, w \in \mathbb{D},$$

for every $f \in \mathcal{A}^p_\sigma$. By Hölder's inequality, with $q = p/(p-1)$,

$$\frac{|f(z)|^p}{|1-z\overline{w}|^p} \leq \int_{\mathbb{D}} \frac{|f(\lambda)|^p dA_\sigma(\lambda)}{|1-\overline{\lambda}z|^{2+\sigma}} \times \left(\int_{\mathbb{D}} \frac{dA_\sigma(\lambda)}{|1-\lambda\overline{w}|^q |1-\lambda\overline{z}|^{(2+\sigma)p}} \right)^{\frac{p}{q}}.$$

Taking $w = z$, and using the standard estimate ([16, Lemma 3.10])

$$\int_{\mathbb{D}} \frac{dA_c(\lambda)}{|1-z\overline{\lambda}|^{2+c+d}} \asymp \frac{1}{(1-|z|^2)^d}, \qquad \text{if } d > 0 \ , \ c > -1, \tag{2}$$

we get the desired conclusion. □

For $\lambda \in \mathbb{D}$, consider the test function

$$F_{\lambda,\beta}(z) = \frac{1}{(1-\overline{\lambda}z)^{1+\beta}}, \qquad z \in \mathbb{D}.$$

If $\beta \geq 0$ is chosen such that $\delta := \delta(p,\alpha,\beta) = 2+\beta-(2+\alpha)/p > 0$, by (2), we have

$$\|F_{\lambda,\beta}\|^p_{\mathcal{D}^p_\alpha} \asymp (1-|\lambda|^2)^{-p\delta}.$$

The following theorem unifies and generalizes the previously known results of MacCluer [3, Theorem 3.12], Tjani [12, Theorem 3.5] and Wriths-Xiao [13, Theorem 3.2] on Hardy, Besov and weighted Dirichlet spaces, respectively.

As mentioned before, the proof we provide here is short and simple.

Theorem 2.2. *Let $p > 1$. Suppose $\varphi \in \mathcal{D}^p_\alpha$ satisfies $\varphi(\mathbb{D}) \subset \mathbb{D}$. Fix $\beta \geq 0$ such that $\delta := \delta(p,\alpha,\beta) = 2+\beta-(2+\alpha)/p > 0$. Then*

(a) C_φ is bounded on $\mathcal{D}^p_\alpha \iff \sup_{\lambda \in \mathbb{D}} (1-|\lambda|^2)^\delta \|F_{\lambda,\beta} \circ \varphi\|_{\mathcal{D}^p_\alpha} < \infty$;

(b) C_φ is compact on $\mathcal{D}^p_\alpha \iff \lim_{|\lambda| \to 1} (1-|\lambda|^2)^\delta \|F_{\lambda,\beta} \circ \varphi\|_{\mathcal{D}^p_\alpha} = 0$.

Proof. To prove (a), we observe that if C_φ is bounded , then

$$\|F_{\lambda,\beta}\circ\varphi\|_{\mathcal{D}^p_\alpha} = O((1-|\lambda|^2)^{-\delta}).$$

For the converse we may assume, without loss of generality, that φ fixes the origin. It follows from Lemma 2.1 that, for $f \in \mathcal{D}^p_\alpha$,

$$\begin{aligned}
&\int_{\mathbb{D}} |\varphi'(z)|^p |f'(\varphi(z))|^p \, dA_\alpha(z)\\
\leq\ & C\int_{\mathbb{D}} |\varphi'(z)|^p \Big(\int_{\mathbb{D}} \frac{|f'(\lambda)|^p}{|1-\overline{\lambda}\varphi(z)|^{(2+\beta)p}}\, dA_{2p+\beta p-2}(\lambda)\Big)\, dA_\alpha(z)\\
=\ & C\int_{\mathbb{D}} |f'(\lambda)|^p \Big((1-|\lambda|^2)^{2p+\beta p-2-\alpha}\int_{\mathbb{D}} \frac{|\varphi'(z)|^p}{|1-\overline{\lambda}\varphi(z)|^{(2+\beta)p}}\, dA_\alpha(z)\Big)\, dA_\alpha(\lambda)\\
=\ & C\int_{\mathbb{D}} |f'(\lambda)|^p (1-|\lambda|^2)^{p\delta} \|(F_{\lambda,\beta}\circ\varphi)'\|^p_{p,\alpha}\, dA_\alpha(\lambda).
\end{aligned}$$

Therefore part (a) follows.

(b) Without loss of generality we assume that $\varphi(0) = 0$. Note that C_φ is compact on $\mathcal{D}^p_\alpha$ if and only if for every bounded sequence $(f_n)_n \subset \mathcal{D}^p_\alpha$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, we have $\|C_\varphi(f_n)\|_{\mathcal{D}^p_\alpha} \to 0$, as $n \to \infty$.

Suppose that C_φ is compact. Since $(1-|\lambda|^2)^\delta F_{\lambda,\beta} \to 0$ uniformly on compact subsets of the unit disc, as $|\lambda| \to 1$, we see that

$$\|C_\varphi(F_{\lambda,\beta})\|_{\mathcal{D}^p_\alpha} = o((1-|\lambda|^2)^{-\delta}).$$

Conversely, assume that $\lim\limits_{|\lambda|\to 1}(1-|\lambda|^2)^\delta \|F_{\lambda,\beta}\circ\varphi\|_{\mathcal{D}^p_\alpha} = 0$. Let $(f_n)_n$ be a bounded sequence of $\mathcal{D}^p_\alpha$ such that $f_n \to 0$ uniformly on compact sets. Since $f'_n \to 0$ uniformly on compact sets, it follows from the proof of part (a) and the hypothesis that, for r close enough to 1,

$$\begin{aligned}
\|C_\varphi(f_n)\|^p_{\mathcal{D}^p_\alpha} - |f_n(0)|^p &\leq \int_{r\mathbb{D}} |f'_n(\lambda)|^p (1-|\lambda|^2)^{p\delta} \|(F_{\lambda,\beta}\circ\varphi)\|^p_{p,\alpha} dA_\alpha(\lambda)\\
&+ \int_{\mathbb{D}\setminus r\mathbb{D}} |f'_n(\lambda)|^p (1-|\lambda|^2)^{p\delta} \|(F_{\lambda,\beta}\circ\varphi)'\|^p_{p,\alpha} dA_\alpha(\lambda) \to 0, \quad n\to\infty
\end{aligned}$$

which finishes the proof. □

The following result is an immediate consequence of Theorem 2.2.

Corollary 2.3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ such that $\varphi \in \mathcal{D}$.*

(a) If $\sup\limits_{n\geq 1} \mathcal{D}(\varphi^n) < \infty$, then C_φ is bounded;

(b) If $\lim\limits_{n\to\infty} \mathcal{D}(\varphi^n) = 0$, then C_φ is compact.

Proof. We consider the test function $F_{\lambda,0}$ with $\beta = \alpha = 0$ and $p = 2$. Both (a) and (b) follow from the following inequality:

$$\begin{aligned}
\mathcal{D}(F_{\lambda,0}\circ\varphi) &\leq 2\,(1-|\lambda|^2)^2 \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\lambda|^2\varphi(z)|^2)^4}\,dA(z)\\
&\leq c\,(1-|\lambda|^2)^2 \sum_{n\geq 0}(n+1)^3\,|\lambda|^{2n}\int_{\mathbb{D}}|\varphi'(z)|^2|\,\varphi^n(z)|^2\,dA(z)\\
&= c\,(1-|\lambda|^2)^2\sum_{n\geq 0}(1+n)\,|\lambda|^{2n}\,\mathcal{D}(\varphi^{n+1})\\
&\leq c\,\limsup_{n\to\infty}\mathcal{D}(\varphi^{n+1}).
\end{aligned}$$

□

Remarks 2.4.

1. The compactness criterion for C_φ in the Bloch space is equivalent to $\|\varphi^n\|_{\mathcal{B}} \to 0$ as was shown in [15] (see also [11, 12]). In the case of the Hardy space H^2, however, we know that if C_φ is compact on H^2 then $\|\varphi^n\|_{\mathrm{H}^2} \to 0$ but the converse does not hold [3]. Note that as before in the proof of Corollary 2.3 ($\beta = 0, \alpha = 1$ and $p = 2$) if $\|\varphi^n\|_{\mathrm{H}^2} = o(1/\sqrt{n})$, then C_φ is compact on H^2.

2. The characterization of compact composition operators on the Dirichlet space in terms of Carleson measures can be found in [3, 12, 17]. A positive Borel measure μ given on $\mathbb{D}$ satisfying

$$\int_{\mathbb{D}} |f(z)|^2\,d\mu(z) \leq \|f\|^2_{2,0}, \qquad f \in \mathcal{A}^2_0,$$

is called a Carleson measure for $\mathcal{A}^2_0$, i.e., the identity map $i_0 : \mathcal{A}^2_0 \to \mathrm{L}^2(\mu)$ is a bounded operator. Such a measure has the following equivalent properties (see [16, Theorem 7.4]). A positive Borel measure μ is a Carleson measure for $\mathcal{A}^2_0$ if and only if

$$\sup_{\lambda\in\mathbb{D}}(1-|\lambda|^2)^2\int_{\mathbb{D}}\frac{d\mu(z)}{|1-\overline{\lambda}z|^4} < \infty,$$

or, equivalently,

$$\sup_{I\subset\mathbb{T}}\ \mu(S(I))/|I|^2 < \infty,$$

for any subarc $I \subset \mathbb{T}$ with arclengh $|I|$, and $S(I)$ is the Carleson box.

The measure μ is called vanishing (or compact) Carleson measure for $\mathcal{A}^2_0$ if the identity map $i_\alpha : \mathcal{A}^2_0 \to \mathrm{L}^2(\mu)$ is a compact operator. This happens if and only if

$$\lim_{|\lambda|\to 1}(1-|\lambda|^2)^2\int_{\mathbb{D}}\frac{d\mu(z)}{|1-\overline{\lambda}z|^4}=0 \iff \lim_{|I|\to 0}\ \mu(S(I))/|I|^2=0.$$

Let $\varphi:\mathbb{D}\to\mathbb{D}$ be analytic and denote by $n_\varphi(z)$ the multiplicity of φ at z. By the change of variable formula [10],

$$\|F_{\lambda,0}\|_{2,0}^p=(1-|\lambda|^2)^2\int_{\mathbb{D}}\frac{n_\varphi(z)\,dA(z)}{|1-\overline{\lambda}z|^4}.$$

Therefore, as a consequence of Theorem 2.2, C_φ is bounded in $\mathcal{D}$ if and only if $n_\varphi(z)dA(z)$ is a Carleson measure for $\mathcal{A}_0^2$ and C_φ is compact in $\mathcal{D}$ if and only if $n_\varphi(z)dA(z)$ is a vanishing Carleson measure for $\mathcal{A}_0^2$. More explicitly, we have

$$\begin{cases} C_\varphi \quad \text{is bounded in } \mathcal{D} & \iff \displaystyle\sup_{I\subset\mathbb{T}}\frac{1}{|I|^2}\int_{S(I)}n_\varphi(z)\,dA(z)<\infty;\\ C_\varphi \quad \text{is compact in } \mathcal{D} & \iff \displaystyle\lim_{|I|\to 0}\frac{1}{|I|^2}\int_{S(I)}n_\varphi(z)\,dA(z)=0.\end{cases}$$

3. Hilbert-Schmidt membership

In the case of the Hardy space H^2, one can completely describe the membership of C_φ in the Hilbert-Schmidt class in terms of the size of the level sets of the inducing map φ. Indeed, C_φ is Hilbert-Schmidt in H^2 if and only if

$$\sum_{n\geq 0}\|\varphi^n\|_{\mathrm{H}^2}^2=\int_{\mathbb{T}}\frac{|d\zeta|}{1-|\varphi(\zeta)|^2}<\infty.$$

Given an arbitrary measurable function f on $\mathbb{T}$, consider the associated distribution function m_f defined by

$$m_f(\lambda)=|\{\zeta\in\mathbb{T}:\ |f(\zeta)|>\lambda|\},\qquad \lambda>0.$$

It then follows that C_φ is in the Hilbert-Schmidt class of H^2 if and only if

$$\int_{\mathbb{T}}\frac{|d\zeta|}{1-|\varphi(\zeta)|^2}=\int_1^\infty m_{(1-|\varphi|^2)^{-1}}(\lambda)\,d\lambda\asymp\int_0^1\frac{|E_\varphi(s)|}{(1-s)^2}\,ds<\infty.$$

It was shown by Gallardo–González [8, Theorem] that there is a mapping φ taking $\mathbb{D}$ to itself such that C_φ is compact in H^2, and that the level set $E_\varphi(1)$ has Hausdorff measure equal to one. Recall that the Hausdorff dimension of E

$$d(E)=\inf\{\alpha:\ \Lambda_\alpha(E)=0\}$$

where $\Lambda_\alpha(E)$ is the α–Hausdorff measure of E given by

$$\Lambda_\alpha(E) = \lim_{\epsilon\to 0}\inf\Big\{\sum_i |\Delta_i|^\alpha : E \subset \bigcup_i \Delta_i\ ,\ |\Delta_i| < \epsilon\Big\}.$$

Given $E \subset \mathbb{T}$ and $t > 0$, let us write $E_t = \{\zeta : d(\zeta, E) \le t\}$ where d denotes the arclength distance and $|E_t|$ denotes the Lebesgue measure of E.

Let E be a closed subset of $\mathbb{T}$ with $|E_t| = O((\log(e/t))^{-3})$ and E has Hausdorff dimension one. (such examples can be given by generalized Cantor sets [2]). Let $\omega(t) = (\log(e/t))^{-2}$, and consider the outer function given by

$$|f_{\omega,E}(\zeta)| = e^{-w(d(\zeta,E))}, \qquad \text{a.e on } \mathbb{T}.$$

Since ω satisfies the Dini condition

$$\int_0 \frac{\omega(t)}{t} dt < \infty,$$

it follows that $f_{\omega,E} \in A(\mathbb{D}) := \mathrm{Hol}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$, disc algebra (see [9] p.105–106) and so $E_{f_{\omega,E}}(1) = E$. On the other hand

$$\int_{\mathbb{T}} \frac{|d\zeta|}{1 - |f_{\omega,E}(\zeta)|^2} \asymp \int_{\mathbb{T}} \frac{|d\zeta|}{\omega(d(\zeta,E))} \asymp \int_0 |E_t| \frac{\omega'(t)}{\omega(t)^2} dt,$$

(see [4, Proposition A.1] for the last equality). Since the last integral converges, C_φ is a Hilbert-Schmidt operator in H^2.

We have the following more precise result.

Theorem 3.1. *Let E be a closed subset of $\mathbb{T}$ with Lebesgue measure zero. There exists a mapping $\varphi : \mathbb{D} \to \mathbb{D}$, $\varphi \in A(\mathbb{D})$ such that C_φ is a Hilbert-Schmidt operator on H^2 and that $E_\varphi(1) = E$.*

Proof. The proof is based a well known construction of peak functions in the disc algebras. Let $\mathbb{T} \setminus E = \cup_{n\ge 1}(e^{ia_n}, e^{ib_n})$. For $t \in (a_n, b_n)$, we define

$$g(e^{it}) = \tau_n \frac{(b_n - a_n)^{1/2}}{((b_n - a_n)^2 - (2t - (b_n + a_n))^2)^{1/4}},$$

where $(\tau_n)_n \subset (0, \infty)$ will be chosen later, and $g(e^{it}) := +\infty$ if $e^{it} \in E$. Note that

$$\int_0^{2\pi} g(e^{it})^2\, dt = \pi \sum_{n=1}^{\infty} \tau_n^2\, (b_n - a_n).$$

Since $\sum_{n=1}^{\infty}(b_n - a_n) = 2\pi$, there exists a sequence $(\tau_n)_n$ such that

$$\lim_{n\to+\infty} \tau_n = +\infty \quad \text{and} \quad \sum_{n=1}^{\infty} \tau_n^2\,(b_n - a_n) < \infty.$$

Let U denote the harmonic extension of g on the unit disc given by

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{|e^{it} - re^{i\theta}|^2}\, g(e^{it})\, dt = \sum_{n\in\mathbb{Z}} \widehat{g}(n)\, r^{|n|}\, e^{in\theta}.$$

Since $\tau_n \to \infty$, one can easily verify that $\lim_{t\to\theta} g(e^{it}) = +\infty$, for $e^{i\theta} \in E$. Hence, $\lim_{r\to 1^-} U(re^{i\theta}) = +\infty$, for $e^{i\theta} \in E$.
Let V be the harmonic conjugate of U, with $V(0) = 0$. It is given by

$$V(re^{i\theta}) = \sum_{n\neq 0} \frac{n}{|n|}\,\widehat{g}(n)\, r^{|n|}\, e^{in\theta}.$$

Now, since g is a C^1 function on $\mathbb{T}\setminus E$, we see that the holomorphic function $f = U + iV$ is continuous on $\overline{\mathbb{D}}\setminus E$. Knowing that $\lim_{r\to 1^-} U(re^{it}) = +\infty$, for $e^{it} \in E$, we get that $\varphi = \dfrac{f}{f+1} \in A(\mathbb{D})$, disc algebra, and $E_\varphi(1) = E$. Finally

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} \frac{dt}{1 - |\varphi(e^{it})|^2} &= \frac{1}{2\pi}\int_0^{2\pi} \frac{(U(e^{it})+1)^2 + V^2(e^{it})}{(U(e^{it})+1)^2 - U^2(e^{it})} dt \\
&\le \frac{1}{2\pi}\int_0^{2\pi} (U(e^{it})+1)^2 + V^2(e^{it}) dt \\
&\le 1 + 2\sum_{n\in\mathbb{Z}} |\,\widehat{g}(n)|^2,
\end{aligned}$$

which shows that C_φ is Hilbert-Schmidt because $g \in L^2(\mathbb{T})$. $\square$

Let E be a closed subset of the unit circle $\mathbb{T}$. Fix a non-negative function $w \in C^1(0,\pi]$ such that

$$\int_{\mathbb{T}} w(d(\zeta, E))\,|d\zeta| < \infty,$$

where d denotes the arclength distance. Now, let $f_{w,E}$ be the outer function given by

$$|f^*_{w,E}(\zeta)| = e^{-w(d(\zeta,E))}, \qquad \text{a.e. on } \mathbb{T}. \tag{3}$$

The following lemma gives an estimate for the Dirichlet integral of $f_{w,E}$ in terms of w and the distance function on E. The proof is based on Carleson's formula, and can be achieved by slightly modifying the arguments used in [5, Theorem 4.1].

Lemma 3.2. *Assume that the function ω is nondecreasing and $\omega(t^\gamma)$ is concave for all $\gamma > 2$. Then*

$$\mathcal{D}(f_{w,E}) \asymp \int_{\mathbb{T}} \omega'(d(\zeta,E))^2\, e^{-2w(d(\zeta,E))}\, d(\zeta,E)\, |d\zeta|.$$

Since the sequence $\{z^n/\sqrt{n+1}\}_{n=0}^{\infty}$ is an orthonormal basis of $\mathcal{D}$, the operator C_φ is Hilbert-Schmidt on the Dirichlet space if and only if

$$\frac{1}{\pi}\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}\, dA(z) = \sum_{n\geq 1} \frac{\mathcal{D}(\varphi^n)}{n} \;<\; \infty.$$

Theorem 3.3. *Assume that the function ω is nondecreasing and $\omega(t^\gamma)$ is concave for some $\gamma > 2$. Then $C_{f_{w,E}}$ is in the Hilbert-Schmidt class in $\mathcal{D}$ if and only if*

$$\int_{\mathbb{T}} \frac{\omega'(d(\zeta,E))^2}{w(d(\zeta,E))^2}\, d(\zeta,E)\, |d\zeta| \;<\; \infty.$$

Proof. We first note that $f_{w,E}^n = f_{nw,E}$. Therefore, by Lemma 3.2, we have

$$\begin{aligned}
\int_{\mathbb{D}} \frac{|f'_{w,E}(z)|^2}{(1-|f_{w,E}(z)|^2)^2}\, dA(z) &= \sum_{n=1}^{\infty} \frac{\mathcal{D}(f_{nw,E})}{n} \\
&\asymp \int_{\mathbb{T}} \omega'(d(\zeta,E))^2\, d(\zeta,E) \sum_{n=1}^{\infty} n e^{-2nw(d(\zeta,E))}\, |d\zeta| \\
&\asymp \int_{\mathbb{T}} \frac{\omega'(d(\zeta,E))^2}{[1-e^{-2w(d(\zeta,E))}]^2}\, d(\zeta,E)\, |d\zeta|.
\end{aligned}$$

Since $1 - e^{-2w(d(\zeta,E))} \asymp w(d(\zeta,E))$, the result flows. $\square$

Given a (Borel) probability measure μ on $\mathbb{T}$, we define its α-energy, $0 \leq \alpha < 1$, by

$$I_\alpha(\mu) = \sum_{n=1}^{\infty} \frac{|\widehat{\mu}(n)|^2}{n^{1-\alpha}}.$$

For a closed set $E \subset \mathbb{T}$, its α-capacity $\mathrm{cap}_\alpha(E)$ is defined by

$$\mathrm{cap}_\alpha(E) := 1/\inf\{I_\alpha(\mu) : \mu \text{ is a probability measure on } E\}.$$

If $\alpha = 0$, we simply note $\mathrm{cap}(E)$ and this means the logarithmic capacity of E.

The weak–type inequality for capacity [2] states that, for $f \in \mathcal{D}$ and $t \geq 4\|f\|_{\mathcal{D}}^2$,

$$\operatorname{cap}(\{\zeta : |f(\zeta)| \geq t\}) \leq \frac{16\|f\|_{\mathcal{D}}^2}{t^2}.$$

As a result of this inequality, we see that if $\liminf \|\varphi^n\|_{\mathcal{D}} = 0$, then $\operatorname{cap}(E_\varphi(1)) = 0$. Indeed, since $E_\varphi(1) = E_{\varphi^n}(1)$, the weak capacity inequality implies that

$$\operatorname{cap}(E_\varphi(1)) = \operatorname{cap}(E_{\varphi^n}(1)) \leq 16\|\varphi^n\|_{\mathcal{D}}^2.$$

Now let $n \to \infty$. Hence, in particular, if the operator C_φ is in the Hilbert-Schmidt class in $\mathcal{D}$, then $\operatorname{cap}(E_\varphi(1)) = 0$. This result was first obtained by Gallardo–González [6, 7] using a completely different method. Theorems 3.4 and 3.6 give quantitative versions of this result.

Theorem 3.4. *If C_φ is a Hilbert-Schmidt operator in $\mathcal{D}$, then*

$$\int_0^1 \frac{\operatorname{cap}(E_\varphi(s))}{1-s} \log \frac{1}{1-s}\, ds < \infty. \tag{4}$$

Proof. Fix $\lambda \in \mathbb{T}$ and let

$$\varphi_\lambda(\zeta) = \log \operatorname{Re} \frac{1+\lambda\varphi(\zeta)}{1-\lambda\varphi(\zeta)}, \qquad \zeta \in \mathbb{T}.$$

Since

$$\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)^2|)^2}\, dA(z) < \infty,$$

it follows that $\varphi_\lambda \in \mathcal{D}(\mathbb{T})$, see [6], where

$$\mathcal{D}(\mathbb{T}) := \{f \in L^2(\mathbb{T}) : \|f\|_{\mathcal{D}(\mathbb{T})}^2 = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2(1+|n|) < \infty\}.$$

Setting $\Delta_\lambda := \{\zeta \in \mathbb{T} : |1-\lambda\varphi(\zeta)| \geq 1\}$, we see that

$$|\varphi_\lambda(\zeta)| \asymp \log \frac{1}{1-|\varphi(\zeta)|^2}, \qquad \forall \zeta \in \Delta_\lambda.$$

Applying the strong capacity inequality [14, Theorem 2.2] to φ_λ, we get

$$\begin{aligned}
\infty > \|\varphi_\lambda\|^2_{\mathcal{D}(\mathbb{T})} &\geq c\int^\infty \operatorname{cap}\{\zeta \in \mathbb{T} : |\varphi_\lambda(\zeta)| > s\}\, ds^2\\
&= c\int^\infty \operatorname{cap}\Big\{\zeta \in \mathbb{T} : \Big|\log \frac{1-|\varphi(\zeta)|^2}{|1-\lambda\varphi(\zeta)|^2}\Big| > s\Big\}\, ds^2\\
&\geq c\int^\infty \operatorname{cap}\Big\{\zeta \in \mathbb{T}\cap\Delta_\lambda : \Big|\log \frac{1-|\varphi(\zeta)|^2}{|1-\lambda\varphi(\zeta)|^2}\Big| > s\Big\}\, ds^2\\
&\geq c\int^\infty \operatorname{cap}\Big\{\zeta \in \mathbb{T}\cap\Delta_\lambda : \log \frac{1}{1-|\varphi(\zeta)|^2} > 4s\Big\}\, ds^2\\
&\geq c_1\int^1 \operatorname{cap}\Big\{\zeta \in \mathbb{T}\cap\Delta_\lambda : |\varphi(\zeta)| > u\Big\}\, d\big(\log\frac{1}{1-u}\big)^2.
\end{aligned}$$

Since $\mathbb{T} = \Delta_1 \cup \Delta_{-1}$, the subadditivity of the capacity implies that

$$\begin{aligned}
&\infty > \|\varphi_1\|^2_{\mathcal{D}(\mathbb{T})} + \|\varphi_{-1}\|^2_{\mathcal{D}(\mathbb{T})} \geq\\
&\qquad\qquad c_2\int^1 \operatorname{cap}\Big\{\zeta \in \mathbb{T} : |\varphi(\zeta)| > u\Big\}\, d\big(\log\frac{1}{1-u}\big)^2,
\end{aligned}$$

and hence the theorem follows. □

Remarks 3.5.

Since $\{z^n/(1+n)^{\frac{1-\alpha}{2}}\}_{n=0}^\infty$ is an orthonormal basis in $\mathcal{D}_\alpha$, $\alpha \in (0,1)$, C_φ is a Hilbert-Schmidt operator in $\mathcal{D}_\alpha$ if and only if

$$\sum_{n=1}^\infty \frac{\mathcal{D}_\alpha(\varphi^n)}{n^{1-\alpha}} \asymp \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2+\alpha}}\, dA_\alpha(z) < \infty.$$

Therefore, for fixed $\lambda \in \mathbb{T}$, the function

$$\varphi_\lambda(\zeta) = \Big(\operatorname{Re}\frac{1+\lambda\varphi(\zeta)}{1-\lambda\varphi(\zeta)}\Big)^{-\alpha/2}, \qquad (\zeta \in \mathbb{T}),$$

belongs to the weighted harmonic Dirichlet space

$$\mathcal{D}_\alpha(\mathbb{T}) := \{f \in L^2(\mathbb{T}) : \|f\|^2_{\mathcal{D}_\alpha(\mathbb{T})} = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2(1+|n|)^{1-\alpha} < \infty\},$$

(see [7]) . Applying again the strong capacity inequality [14, Theorem 2.2] for $\mathcal{D}_\alpha$ to φ_λ, we get as before

$$\int_0^1 \frac{\operatorname{cap}_\alpha(E_\varphi(s))}{(1-s)^{1+\alpha}}\, ds < \infty.$$

The following theorem is the analogue of Proposition 3.1 for the Dirichlet space. It shows that condition (4) is optimal.

Theorem 3.6. *Let* $h : [1,+\infty[\to [1,+\infty[$ *be a function such that* $\lim_{x\to\infty} h(x) = +\infty$. *Let* E *be a closed subset of* $\mathbb{T}$ *such that* $\operatorname{cap}(E) = 0$. *Then there is* $\varphi \in A(\mathbb{D}) \cap \mathcal{D}$, $\varphi(\mathbb{D}) \subset \mathbb{D}$ *such that :*

(1) $E_\varphi(1) = E$;
(2) C_φ *is in the Hilbert–Schmidt class in* $\mathcal{D}$;
(3) $\displaystyle\int^1 \frac{\operatorname{cap}(E_\varphi(s))}{1-s} \log\frac{1}{1-s} h\Big(\frac{1}{1-s}\Big)\, ds = +\infty.$

Proof. Let $k(x) = h(e^x)$, there exists a continuous decreasing function ψ such that

$$\int^{+\infty} \psi(x)\, dx^2 < \infty \quad \text{and} \quad \int^{+\infty} \psi(x)\, k(x)\, dx^2 = \infty.$$

Set $\eta(t) = \psi^{-1}(\operatorname{cap}(E_t))$. We have

$$\begin{aligned} \int_0 \operatorname{cap}(E_t)\, |d\eta^2(t)| &\asymp \int_0 \psi(\eta(t))\, |d\eta^2(t)| \\ &\asymp \int^{+\infty} \psi(x)\, dx^2 < \infty, \end{aligned}$$

and,

$$\begin{aligned} \int_0 \operatorname{cap}(E_t)\, h(e^{\eta(t)}))\, |d\eta^2(t)| &\asymp \int_0 \psi(\eta(t))\, k(\eta(t))\, |d\eta^2(t)| \\ &\asymp \int^{+\infty} \psi(x)\, k(x)\, dx^2 = \infty. \end{aligned}$$

Since

$$\int_0 \operatorname{cap}(E_t)\, |d\eta^2(t)| < \infty,$$

by [4, Theorem 5], there exists a function $f \in \mathcal{D}$ such that

$$\operatorname{Re} f(\zeta) \geq \eta(d(\zeta, E)) \quad \text{and} \quad |\operatorname{Im} f(\zeta)| < \pi/4, \qquad \text{q.e. on} \quad \mathbb{T}.$$

By harmonicity,

$$|\operatorname{Im} f(z)| < \pi/4\,, \qquad |z| < 1,$$

Now take

$$\varphi = \exp(-e^{-f}).$$

By a simple modification in the construction of f as in [1], we can suppose that $\varphi \in A(\mathbb{D})$. Hence $E_\varphi(1) = E$ and

$$\begin{aligned}\int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} &\asymp \int_{\mathbb{D}} \frac{|f'(z)|^2\, e^{-2\,\mathrm{Re} f(z)}\, e^{-2\,e^{-\mathrm{Re} f(z)}\cos(\mathrm{Im} f(z))}}{e^{-2\,\mathrm{Re} f(z)}\,\cos^2(\mathrm{Im} f(z))}\, dA(z) \\ &\leq \int_{\mathbb{D}} |f'(z)|^2 \exp(-\sqrt{2}\, e^{-\mathrm{Re} f(z)})\, dA(z) \\ &\leq c \int_{\mathbb{D}} |f'(z)|^2\, dA(z) < \infty.\end{aligned}$$

Hence C_φ is in the Hilbert–Schmidt class. Finally, since

$$E_\varphi(s) \supseteq \{\zeta \in \mathbb{T} : \eta(d(\zeta, E)) \geq \log(1/1-s)\},$$

we get

$$\int_0 \mathrm{cap}(E_\varphi(s)) h(1/1-s)\, d\big(\log(1/1-s)\big)^2 \geq \int_0 \mathrm{cap}(E_t) h(e^{\eta(t)})\, |d\eta^2(t)| = +\infty.$$

$\square$

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